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Weierstrass Points on Families of Graphs

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Weierstrass Points on Families of Graphs

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Chapter 1
Introduction

In this chapter we provide a brief introduction to the main content of the thesis. Formal definitions and examples are presented in Chapter 2. Our original work regarding Weierstrass points on complete graphs and complete bipartite graphs is presented in Chapter 3. We also introduce the notion of concentrating a divisor on a single vertex. In Chapter 4, we review the recent work of Baker and Norine regarding Harmonic Morphisms on Hyperelliptic Graphs. The first theorem we present confirms the ubiqitous presence of Weierstrass points on complete graphs.

1.0.1 Theorem. Each vertex in the complete graph K_n with $n \geq 4$ vertices is a Weierstrass point.

We then approach the question of Weierstrass points on complete bipartite graphs, which are slightly more complicated than complete graphs. Complete bipartite graphs are defined in Chapter 3.

1.0.2 Theorem. A vertex x in the complete bipartite graph $K_{n,m}$ is a Weierstrass point if and only if the number of vertices on the side of the graph where x is located is greather than 2.

In proving these theorems about Weierstrass points, we realized that it was occasionally possible to move many more than g chips to a given vertex. In some cases, the entire canonical divisor could be concentrated on a single vertex. Although we do not not state them here, we present these theorems and the relevant background in Chapter 3.

Chapter 2

Divisors on Graphs and the Chip Firing Game

In this this chapter we present the notion of a divisor on a graph. We discuss the set of divisors on a graph as well as various subsets of divisors possessing special properties. We give some basic graph theoretic definitions but refer the reader to [] for a more detailed presentation. We follow the notation of Baker and Norine where possible, defining Weierstrass points in terms of linear equivalence and canonical divisors. We also describe the more intuitive Chip Firing Game, highlighting its similarity to the notion of linear equivalence of divisors.

2.1 Basic Definitions

2.1.1 Definition. We define a **graph** G to be a finite, undirected multigraph with vertex set $V(G)$ and edge set $E(G)$. In this thesis, we assume that all graphs are connected and without self loops (an edge that starts and ends at the same vertex). An edge $e \in E(G)$ may be seen as an unordered pair of vertices, $e = (x, y)$. Each undirected edge will be treated as two directed edges pointing opposite ways.

2.1.2 Definition. Let G be a graph and let $x \in V(G)$. We define the **edge set** of

x, denoted E_x to be all $e \in E(G)$ such that $x \in e$. The cardinality of the edge set of x is called the **degree** of x and is denoted d_x.

2.1.3 Definition. A **divisor** D on a graph G is an integer vector indexing the vertices of G. $D(x)$ will denote the value of D at the vertex $x \in V(G)$. The set of all divisors on G, which can be seen as the free abelian group over V(G), will be denoted by Div(G).

2.1.4 Definition. We say that a divisor $D \in \text{Div}(G)$ is an **effective divisor** if $D(x) \geq 0$ for all $x \in V(G)$. In this case, we write $D \geq 0$. The set of effective divisors on G is denoted by $\text{Div}^+(G)$.

2.1.5 Definition. We define the **degree function** deg: $\text{Div}(G) \to \mathbb{Z}$ by

$$\deg(D) = \sum_{x \in V(G)} D(x),$$

where $D \in \text{Div}(G)$. The image of the degree function at a given divisor D is called the **degree** of D.

2.1.6 Definition. The subset of divisors on G having degree 0 is a subgroup of Div(G) and is denoted $\text{Div}_0(G)$. More generally, we denote the set of divisors on G of degree k by $\text{Div}_k(G)$.

2.1.7 Definition. Let G be a graph. We define the **canonical divisor** on G, denoted C_G, as follows:

$$C_G = \sum_{x \in V(G)} (d_x - 2)(x).$$

2.1.8 Example. In Figure 2.1.1, we present the house-X graph. The canonical divisor in this case is $[1, 1, 2, 2, 0]$ and the degree of the canonical divisor is 6. We will see in Proposition 2.1.12 that the degree of the canonical divisor depends directly on structure of the graph.

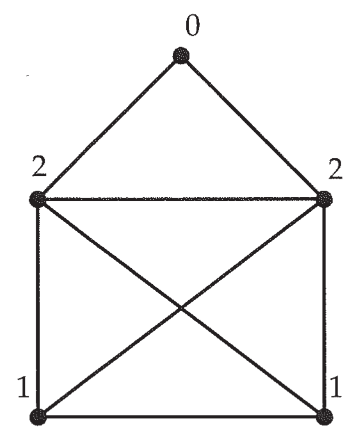

Figure 2.1.1: House X Graph and its Canonical Divisor

2.1.9 Definition. Let G be a graph. A **cycle** in G is defined to be any closed path which does not repeat vertices or edges.

2.1.10 Definition. We define the **genus** of a graph G by the formula

$$g(G) = |E(G)| - |V(G)| + 1.$$

When the identity of the graph is clear from context, we will write g, as opposed to $g(G)$.

2.1.11 Remark. As defined above, the genus of a graph G is its cyclotomic number, or the minimum number of edges that must be removed from G to make it cycle-free.

2.1.12 Proposition. Let G be a graph with genus g. Then, $\deg(C_G) = 2g - 2$.

Proof. By definition of the canonical divisor, we know that

$$\begin{aligned} \deg(C_G) &= \sum_{x \in V(G)} (d_x - 2) \\ &= \sum_{x \in V(G)} d_x - \sum_{x \in V(G)} 2 \\ &= 2|E(G)| - 2|V(G)| \\ &= 2(|E(G) - |V(G)| + 1) - 2 \\ &= 2g - 2. \end{aligned}$$

□

2.2 Linear Equivalence of Divisors

We will now define an equivalence relation between divisors. This equivalence relation, called linear equivalence, relies on a few other concepts, such as functions on graphs and divisors of functions, which we will now make precise.

2.2.1 Definition. Let $C^0(G, \mathbb{Z})$ be the set of all $\mathbb{Z}$-valued functions on $V(G)$. The

divisor of a function $f \in C^0(G, \mathbb{Z})$ is defined by the following formula:

$$\operatorname{div}(f) = \sum_{x \in V(G)} \left(\sum_{(x,y) \in E_x} f(x) - f(y) \right) (x).$$

2.2.2 Definition. Let G be a graph. Divisors on G that can be expressed in the form $\operatorname{div}(f)$ where $f \in C^0(G, \mathbb{Z})$ are called **principal divisors**. The set of principal divisors on G is denoted $\operatorname{Prin}(G)$.

It should be quite clear that any divisor in $\operatorname{Prin}(G)$ has degree 0. More interestingly however, the elements of $\operatorname{Prin}(G)$ along with vertex-wise addition form a subgroup of $\operatorname{Div}(G)$.

2.2.3 Proposition. $\operatorname{Prin}(G)$ is a subgroup of $\operatorname{Div}_0(G)$.

Proof. Let $D \in \operatorname{Prin}(G)$. Then there exists $f \in C^0(G, \mathbb{Z})$ such that $\operatorname{div}(f) = D$. That is to say

$$D = \operatorname{div}(f) = \sum_{x \in V(G)} \left(\sum_{(x,y) \in E_x} f(x) - f(y) \right) (x).$$

In the above summation, each edge $e \in \mathrm{E(G)}$ appears exactly twice (once for each of its endpoints). Thus, for each vertex $w \in V(G)$, $f(w)$ and $-f(w)$ appear the same number of times, canceling each other out. Therefore, the overall summation equals 0 and $D \in \operatorname{Div}_0(G)$.

To show that $\operatorname{Prin}(G)$ is a subgroup of $\operatorname{Div}_0(G)$, it suffices to show that if $D_3 = D_1 - D_2$, where D_1 and D_2 are both members of $\operatorname{Prin}(G)$, then $D_3 \in \operatorname{Prin}(G)$. Let

$f_1, f_2 \in C^0(G, \mathbb{Z})$ such that $\operatorname{div}(f_1) = D_1$ and $\operatorname{div}(f_2) = D_2$. Thus,

$$\begin{aligned}
D_3 = D_1 - D_2 &= \operatorname{div}(f_1) - \operatorname{div}(f_2) \\
&= \sum_{x \in V(G)} \left(\sum_{(x,y) \in E_x} f_1(x) - f_1(y) \right)(x) - \sum_{x \in V(G)} \left(\sum_{(x,y) \in E_x} f_2(x) - f_2(y) \right)(x) \\
&= \sum_{x \in V(G)} \left(\sum_{(x,y) \in E_x} (f_1(x) - f_2(x)) - (f_1(y) - f_2(y)) \right)(x) \\
&= \sum_{x \in V(G)} \left(\sum_{(x,y) \in E_x} f_3(x) - f_3(y) \right)(x) \\
&= \operatorname{div}(f_3)
\end{aligned}$$

where $f_3 = f_1 - f_2$ and is clearly a member of $C^0(G, \mathbb{Z})$. It follows that $\operatorname{Prin}(G)$ is a subgroup of $\operatorname{Div}_0(G)$. □

It is clear from the above proposition that any divisor of non-zero degree cannot be written in the form $\operatorname{div}(f)$. It is less clear that degree zero divisors that are not principal actually exist. This fact, however, is crucial to the thesis. If $\operatorname{Prin}(G) = \operatorname{Div}_0(G)$, all divisors of the same degree would be linearly equivalent. If this were true, the whole notion of linear equivalence would be relatively trivial.

2.2.4 Example. The divisor $[0, 0, 0, 1, -1]$ on the graph depicted in the figure below has degree 0 but is not a prinipal divisor. We suggest that the reader confirm this by searching for a firing sequence that results in an effective divisor. We will prove this fact rigorously in Chapter 3.

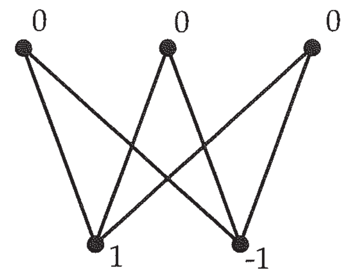

Figure 2.2.1: A Non-principal Divisor

2.2.5 Definition. We define an equivalence relation $\sim$ on $\mathrm{Div}(G)$ called **linear equivalence**. We say $D \sim D'$ if and only if $D - D' \in \mathrm{Prin}(G)$. Given a divisor D, the **complete linear system** associated to D is defined as the set of all effective divisors equivalent to D. This complete linear system is denoted $|D|$.

It turns out that the equivalence classes of $\mathrm{Div}_0(G)$ under linear equivalence form an additive group. This group, called the Jacobian of a graph is defined below.

2.2.6 Definition. Let G be a graph and let $\mathrm{Div}_0(G)$ denote the set of degree 0 divisors on G and let $\mathrm{Prin}(G)$ denote the set of principal divisors on G. The **Jacobian** of G, denoted $\mathrm{Jac}(G)$ is defined

$$\mathrm{Jac}(G) = \mathrm{Div}_0(G)/\mathrm{Prin}(G).$$

Using Kirchhoff's Matrix-Tree Theorem ([1], §14), it can be shown that $\mathrm{Jac}(G)$ is a finite abelian group of order κ_G, where κ_G is the number of spanning trees in G. We are led to the following simple lemma, which is proven in detail in [3].

2.2.7 Lemma. Let G be a graph. We have $(x) \sim (y)$ for all x and y in $V(G)$ if and only if G is a tree.

We now introduce The Chip Firing Game which will provide us with an intuitive framework for computations and proofs regarding linear equivalence of divisors. First we define a chip configuration to be a divisor. We then define an analogue of linear equivalence that relies on discrete movements of chips called firings. This system of chip configurations and equivalence by firing exactly matches our notion of linear equivalence of divisors and is far easier to visualize. Lastly, we reformulate our definition of the Jacobian of a graph in terms of chip firing.

2.2.8 Definition. Let G be a graph. A vertex $x \in V(G)$ can **fire**, sending exactly one chip along each of its edges to neighboring vertices, decreasing its chip count by its degree, d_x. A vertex can also **reverse fire**, stealing one chip from each neighbor and increasing its chip count by its degree, d_x.

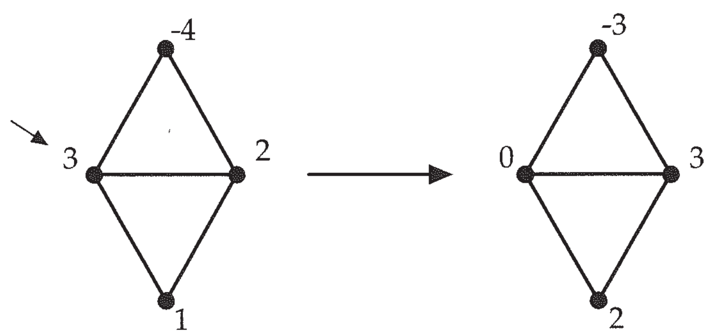

Figure 2.2.2: Example of Chip Firing

2.2.9 Remark. It is important to note that we do not impose any limits on which vertices can be fired or reverse fired. Some versions of the Chip Firing Game forbid firings or reverse firings that result in negative chip counts at individual vertices. In our version, negative chip counts are allowed.

Using machinery from linear algebra, we can more rigorously describe the process of chip firing. The firing or reverse firing of a vertex corresponds to an addition or subtraction of a row vector from the graph's Laplacian matrix, defined below.

2.2.10 Definition. Let G be a graph and let $V(G) = \{x_1, x_2, x_3, \ldots x_n\}$.The **adjacency matrix** of G, denoted A_G is the $n \times n$ matrix where the (i, j)-entry is the number of edges between x_i and x_j and all other entries are 0.

2.2.11 Definition. Let G be a graph and let $V(G) = \{x_1, x_2, x_3, \ldots x_n\}$.The **degree matrix** of G, denoted D_G, is the $n \times n$ matrix where the (i, i)-entry is the degree of x_i for all $i \in [0, n]$ and all other entries are 0.

2.2.12 Definition. Let G be a graph and let $V(G) = \{x_1, x_2, x_3, \ldots x_n\}$. The **Laplacian matrix** of G, denoted Δ_G, is the $n \times n$ matrix defined by

$$\Delta_G = A_G - D_G.$$

We denote the i^{th} row of the Laplacian by Δ_i. Therefore,

$$\Delta_G = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{pmatrix}.$$

2.2.13 Example. The Laplacian matrix of the graph G depicted below is computed as follows

$$\Delta_G = A_G - D_G$$

$$\Delta_G = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

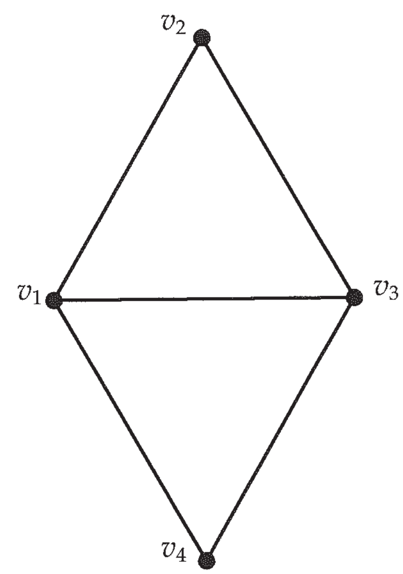

Figure 2.2.3: Laplacian Matrix of a Graph

The i^{th} row of the Laplacian matrix corresponds to the chip change that occurs when firing the i^{th} vertex. Reverse firing the i^{th} vertex is the same as subtracting the i^{th} row of the Laplacian matrix. Now we define a new equivalence relation between chip configurations based on firing and reverse firing.

2.2.14 Definition. Two chip configurations C and C' on a graph G are said to be equivalent by firing if C' can be reached from C by firing or reverse firing a subset of $V(G)$, allowing for repeated firings and reverse firings of a single vertex. When C and C' are equivalent by firing, we write $C \sim_F C'$. In the language of linear algebra, two configurations C and C' are equivalent by firing if $C - C' \in \mathbb{Z}^n\, \Delta_G$, where $\mathbb{Z}^n\, \Delta_G$ denotes the integer row span of Δ_G.

2.2.15 Lemma. The notions of linear equivalence on divisors and equivalence by firing on chip configurations are equivalent.

Proof. Let D and D' be divisors on a graph G and let C and C' be the analagous chip configurations on G. Furthermore, let

$$\Delta_G = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{pmatrix}$$

be the Laplacian on G. Assume $D \sim D'$. We know by definition that

$$D - D' = \operatorname{div}(f)$$

for some function $f \in C^0(G, \mathbb{Z})$. Let f_i be the value of f at the vertex i so that

$f = (f_1, f_2, f_3, \ldots, f_n)$. Now observe that

$$C - C' = D - D' = f_1\Delta_1 + f_2\Delta_2 + f_3\Delta_3 + \ldots + f_n\Delta_n$$

This means that $C - C'$ is an element of the integer row span of Δ_G, and we can concluded that $C \sim_F C'$.

Now suppose that $C \sim_F C'$. Then $C - C'$ is an element of the integer row span of Δ_G. Call this element F and represent it in terms of the row vectors of Δ_G as follows

$$F = f_1\Delta_1 + f_2\Delta_2 + f_3\Delta_3 + \ldots + f_n\Delta_n.$$

Let $f = (f_1, f_2, f_3, \ldots, f_n)$ and observe that $D - D' = \mathrm{div}(f)$. Therefore, $D \sim D'$. $\square$

Now that we have shown that the notions of equivalence of chip configurations by firing and linear equivalence of divisors are in fact themselves equivalent, we redefine the Jacobian of a graph in terms of the Laplacian matrix.

2.2.16 Definition. Let G be a graph and let $\mathbb{Z}^n\,\Delta_G$ denote the integer row span of the Laplacian of G. The **Jacobian** of G, defined initially in Definition 2.2.6, can be redefined as

$$\mathrm{Jac}(G) = \mathrm{Div}_0(G)/\mathbb{Z}^n\,\Delta_G.$$

2.3 Riemann-Roch for Graphs

We now present an important result from Baker and Norine's investigations of divisors on graphs. First, we define the rank of a divisor which refers to the number of chips that can be removed from a divisor while ensuring the divisor's linear equivalence to an effective divisor. We then present Baker and Norine's extension of the Riemann-Roch theorem. The proof can be found in [2].

2.3.1 Definition. Given a divisor D on a graph G, the **rank** of D, denoted $r(D)$, is defined -1 if $|D| = \emptyset$. Otherwise,

$$r(D) = \max\{k \in \mathbb{Z} \text{ such that } |D - E| \neq \emptyset \text{ for all } E \in \mathrm{Div}_k^+(G)\}$$

The rank of a divisor (chip configuration) can also be thought of as the maximum number of chips that can be subtracted from that configuration (from any combination of vertices) that will result in a configuration that is equivalent by firing to a configuration with non-negative chip count at each vertex. The Riemann-Roch Theorem, proven initially by Riemann in 1857, and later by his student Gustav Roch in 1865, is an important tool in algebraic geometry. Baker and Norine have stated and proven an analogue of this theorem for graphs.

2.3.2 Theorem. ([2], Theorem 1.12: Riemann-Roch for graphs) Let D be a divisor and let C_G be the canonical divisor on a graph G. Then

$$r(D) - r(C_G - D) = \deg(D) + 1 - g.$$

We also present a useful theorem from [2] that will allow us to prove Theorem 4.2.12.

2.3.3 Definition. Let $<$ be a linear ordering on $V(G)$. We define a corresponding divisor $\nu \in \mathrm{Div}(G)$ by

$$\nu = \sum_{x \in V(G)} \left(\left| \{ e = xy \in E(G) \mid y < x \} \right| \right) (x).$$

2.3.4 Theorem. ([2], Theorem 3.3)For every $D \in \mathrm{Div}(G)$, exactly one of the following holds:

(i) $r(D) \geq 0$; or

(ii) $r(\nu - D) \geq 0$ for some divisor ν associated to a linear ordering $<$ of $V(G)$.

These results greatly simplify the task of calculating the rank of a divisor. We will apply them in future chapters.

Chapter 3

Weierstrass Points on Families of Graphs

In this chapter, we define Weierstrass points on graphs. Unlike the classical case of algebraic curves, there exist graphs of genus at least two with no Weierstrass points. In some graphs, every vertex is a Weierstrass point and in others Weierstrass points are located in specific places. In their paper *Harmonic Morphisms and Hyperelliptic Graphs*, Matthew Baker and Seguei Norine completely characterize hyperelliptic graphs with no Weierstrass points. We will review their work in Chapter 4. In his work on the topic, Peter Whalen characterizes exactly which genus 3 graphs have Weierstrass points and which do not. Our goal was characterize where Weierstrass points occur in certain families of graphs. The families we study are complete graphs and complete bipartite graphs.

3.1 Weierstrass Points

3.1.1 Definition. Given a graph G with genus g and canonical divisor C_G, a vertex $x \in V(G)$ is called a **Weierstrass point** if

$$r(C_G - g(x)) \geq 0.$$

In the language of chip firing, a vertex $x \in V(G)$ is a Weierstrass point if there exists a firing sequence from the canonical divisor to an effective divisor where $D(x) = g$. Now we will present two examples to illustrate the concept of Weierstrass points more clearly.

3.1.2 Example. *The House-X Graph* The genus of the house-X graph, depicted with its canonical divisor in Figure 3.1.1 and denoted H_X, is calculated as follows:

$$\begin{aligned} g(H_X) &= |E(G)| - |V(G)| + 1 \\ &= 8 - 5 + 1 \\ &= 4 \end{aligned}$$

Therefore, to show that a vertex $x \in V(H_X)$ is a Weierstrass point, it suffices to provide a sequence of firings or reverse firings from the canonical divisor C_{H_X} to an effective divisor with 4 chips at x. We present firing sequences to show that x_1 and x_5 are Weierstrass points in Figure 3.1.1. In Figure 3.1.2, we do the same for x_2 and x_4. Lastly, in Figure 3.1.3, we show that x_3 is a Weierstrass point.

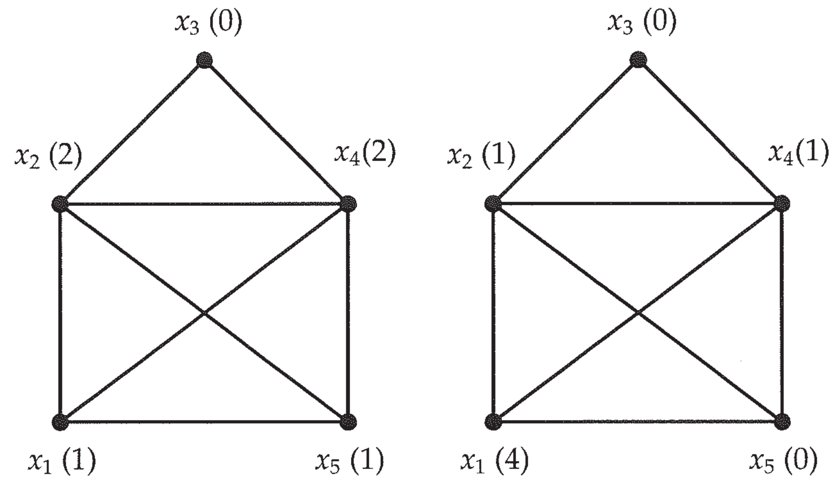

Figure 3.1.1: In the House-X Graph, x_1 is clearly a Weierstrass point. We can move $g = 4$ chips to x_1 by reverse firing x_1 once. This means that x_5 is also a Weierstrass point by symmetry.

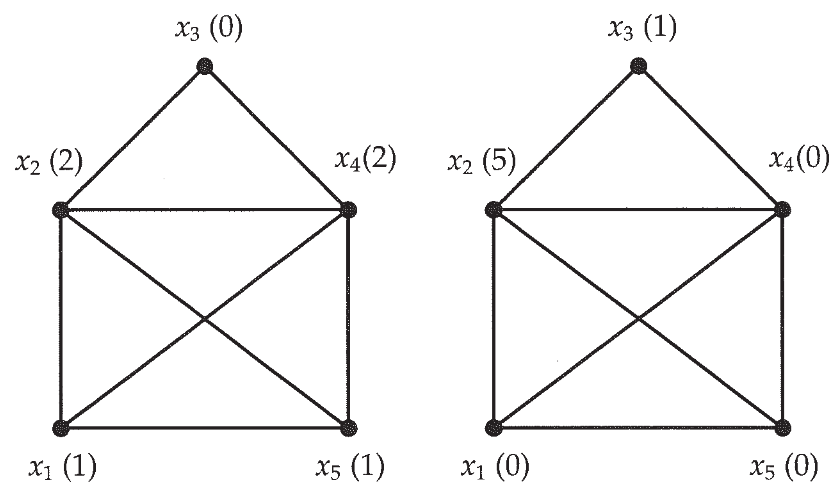

Figure 3.1.2: In the House-X Graph, x_2 is also a Weierstrass point. We can move 5 chips to x_2 by reverse firing x_2 and x_3 each once. This means that x_4 is also a Weierstrass point by symmetry.

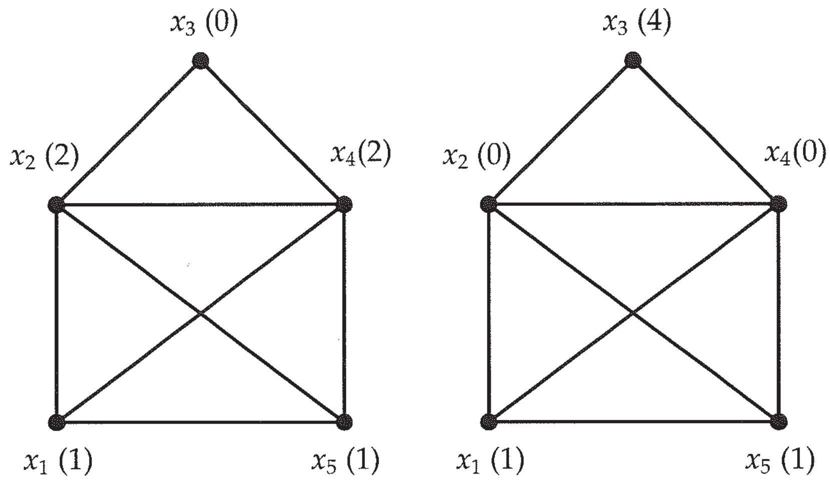

Figure 3.1.3: In the House-X Graph, x_3 is a Weierstrass point. We can move $g = 4$ chips to x_3 by reverse firing x_3 twice.

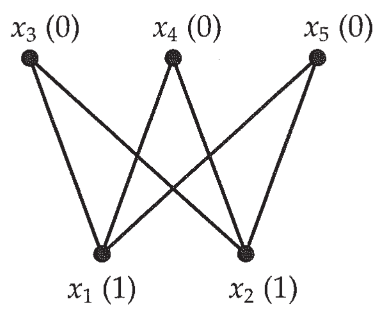

Figure 3.1.4: In the above graph, denoted $K_{2,3}$, vertices x_1 and x_2 are not Weierstrass points. The results we present later in the chapter will confirm this.

3.1.3 Example. *The Complete Bipartite Graph $K_{2,3}$* The genus of this graph, shown in Figure 3.1.4, is given by

$$\begin{aligned} g(K_{2,3}) &= |E(G)| - |V(G)| + 1 \\ &= 6 - 5 + 1 \\ &= 2. \end{aligned}$$

We cannot move $g = 2$ chips to x_1 or x_2, but we can move $g = 2$ chips to all three of the remaining vertices. It is left to the reader to determine appropriate firing sequences.

Lastly, we present an alternative definition of Weierstrass points that follows easily from the Riemann-Roch theorem for graphs (Theorem 2.3.2).

3.1.4 Lemma. Let G be a graph with genus g and let C_G be the canonical divisor on G. A vertex $x \in V(G)$ is a Weierstrass point if and only if $r(g(x)) \geq 1$.

Proof. Recall the statement of Riemann-Roch:

$$r(D) - r(C_G - D) = \deg(D) + 1 - g$$

and let $D = C_G - g(x)$. The resulting formula is

$$r(C_G - g(x)) - r(C_G - (C_G - g(x))) = \deg(C_G - g(x)) + 1 - g \tag{3.1.1}$$

Recall that

$$\deg(C_G) = 2g - 2$$

so

$$\deg(C_G - g(x)) = g - 2.$$

Therefore we can simplify (3.1.1)

$$r(C_G - g(x)) - r(g(x)) = -1$$

which can be rearranged to yield

$$r(C_G - g(x)) = r(g(x)) - 1 \tag{3.1.2}$$

If x is a Weierstrass point, then $r(C_G - g(x)) \geq 0$ which implies by (3.1.2) that $r(g(x)) \geq 1$. Similarly, if $r(g(x)) \geq 1$, it is clear that $r(C_G - g(x)) \geq 0$. □

3.2 Complete Graphs

We now enter our discussion of Weierstrass points on complete graphs.

3.2.1 Definition. A **complete graph** is a graph such that between any two vertices there exists exactly one edge. The complete graph with n vertices is unique and is denoted K_n.

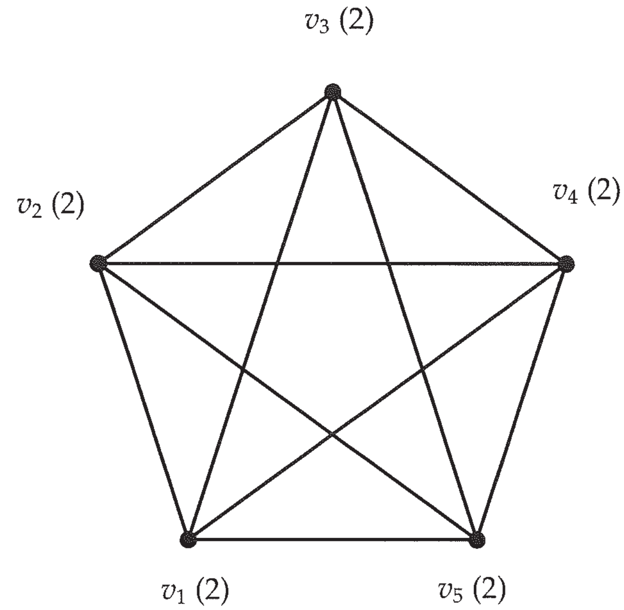

Figure 3.2.1: Complete Graph on 5 Vertices (with its canonical divisor)

3.2.2 Example. Let K_5 denote the complete graph on 5 vertices, presented in Figure 3.2.1. The canonical divisor C_{K_5}, is equal to $[2,2,2,2,2]$ and the genus, g, is given by

$$g = |E(K_5)| - |V(K_5)| + 1 = 10 - 5 + 1 = 6.$$

We will show that all vertices in K_5 are Weierstrass points. Let $x \in V(K_5)$ be arbitrary (all vertices in K_5 are essentially the same by symmetry). From the canonical divisor, we can move 4 chips to x by reverse firing x once (shown in Figure 3.2.1), yielding the divisor $[6,1,1,1,1]$. This divisor is effective and has $g = 6$ chips at x so x is a Weierstrass point. By symmetry, the other vertices are

Weierstrass points as well.

It is natural to extend the idea of Weierstrass points by increasing the amount of chips that need be moved to a vertex while maintaining an effective divisor. In Example 3.2.2, reverse firing x one more time would have located $\deg(C_{K_5}) = 10$ chips at x, leaving all other vertices with zero chips. This is the strongest possible condition we could impose and we formally define it below.

3.2.3 Definition. We say that a divisor D on a graph G can be **concentrated** on a vertex $x \in G$ if $r(D - \deg(D)(x)) \geq 0$.

It turns out that for any complete graph, the canonical divisor can be concentrated to any vertex.

3.2.4 Theorem. The canonical divisor on a complete graph K_n, denoted C_{K_n}, can be concentrated to any vertex in $V(K_n)$.

Proof. Because the complete graph is symmetric, it suffices to prove that C_{K_n} can be concentrated to an arbitrary vertex $x \in V(K_5)$. Consider the canonical divisor on the complete graph of n vertices:

$$C_{K_n} = [n-3, n-3, n-3, \ldots, n-3]$$

Reverse firing x yields the divisor

$$[n-4, n-4, n-4, \ldots, (n-3) + (n-1)].$$

Repeating this reverse firing $n-3$ times results in the configuration:

$$[0,0,0,\ldots,n(n-3)]$$

□

3.2.5 Corollary. Each vertex in a complete graph with $n \geq 4$ vertices is a Weierstrass point.

3.3 Complete Bipartite Graphs

For complete graphs, the question "Where are the Weierstrass points?" has a fairly simple answer: everywhere. Because of their strong inherent symmetry, complete graphs do not offer any structural variety from vertex to vertex. Complete bipartite graphs, on the other hand, have two distinct sets of vertices. The vertices within each set are structurally identical, but no such symmetry exists between vertices in opposing sets. To gain insight into where Weierstrass points occur on graphs, we sought to characterize complete bipartite graphs, which are defined below.

3.3.1 Definition. A **bipartite graph** is a graph that can be partitioned into two disjoint sets U and V such that every edge connects a vertex in U to a vertex in V. A **complete bipartite graph** is a bipartite graph where each vertex in U is connected to every vertex in V exactly once. The number of vertices in U will be denoted by n and the number of vertices in V will be denoted by m.

Similarly, vertices in U will be referred to as **n-side vertices** and those in V will be referred to as **m-side vertices**. The complete bipartite graph with n vertices in U and m vertices in V will be denoted $K_{n,m}$.

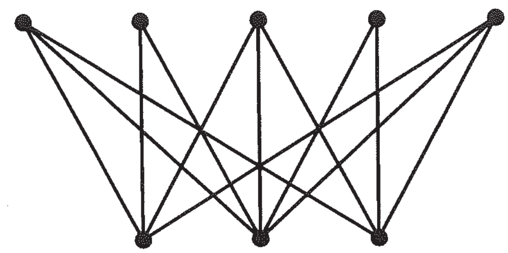

Figure 3.3.1: Complete Bipartite Graph $K_{3,5}$

3.3.2 Example. The genus of the complete bipartite graph $K_{3,5}$, shown in Figure 3.3.1, is given by

$$\begin{aligned} g(K_{n,m}) &= |E(G)| - |V(G)| + 1 \\ &= nm - (n + m) + 1 \\ &= 6 - 5 + 1 \\ &= 2 \end{aligned}$$

Following the example above, we can make the following remark about the genus of a general complete bipartite graph.

3.3.3 Remark. Let $K_{n,m}$ be the complete bipartite graph with n vertices in U and m vertices in V. If either $n = 1$ or $m = 1$, then $g(K_{n,m}) = 0$. This means, for our theorems to make sense, we assume that both n and m are greater than 1.

Before presenting the two main results of this section, we present a general fact about chip firing on undirected graphs.

3.3.4 Lemma. Let G be a graph and let D be an arbitrary divisor on G. Let $x \in V(G)$ be arbitrary as well. Reverse firing x results in the same change to D as firing all vertices in $V(G)$ except x.

Proof. Let x be the k^{th} vertex in $V(G)$ and denote the Laplacian matrix of G by

$$\Delta_G = \begin{pmatrix} \Delta_1 \\ \vdots \\ \Delta_k \\ \vdots \\ \Delta_n \end{pmatrix}.$$

Observe that by the definition of the Laplacian

$$\Delta_1 + \Delta_2 + \ldots + \Delta_k + \ldots + \Delta_n = 0.$$

Therefore,

$$-\Delta_k = \Delta_1 + \Delta_2 + \ldots + \Delta_{k-1} + \Delta_{k+1} + \ldots + \Delta_n.$$

The lemma follows directly from the above statement $\square$

Lemma 3.3.4 simplifies the proofs in this section greatly. Now, instead of incorporating both firing and reverse firing, we can limit the Chip Firing Game, allowing only firing without limiting the scope of our results (since reverse firing one vertex is the same as firing all the others).

3.3.5 Theorem. Let $K_{n,m}$ be the complete bipartite graph with n vertices in U and m vertices in V and assume $n, m > 1$. Then:

(i) A vertex $x \in U$ is a Weierstrass point if and only if $|U| = n \geq 3$.

(ii) A vertex $y \in V$ is a Weierstrass point if and only if $|V| = m \geq 3$.

Proof. We will prove (i) as (ii) follows directly by interchanging U and V and n and m. Let $x \in U$ and let $|U| = n \geq 3$. Observe that by reverse firing x, all of the chips in V can be moved to x without changing the chip counts of any other vertices in U. The number of chips in V is equal to $(n-2)m$. The number of chips already present at x is equal to $m - 2$. Therefore, we are guarenteed that at least $(n-2)m + (m-2)$ chips can be accumulated at x. If this number is greater than $g = nm - (n+m) + 1$, then x is a Weierstrass point. We can see that

$$(n-2)m + (m-2) \geq g = nm - (n+m) + 1$$

Exactly when

$$n \geq 3.$$

If $|U| = n < 3$, we know that $n = 2$, because we assume $n > 1$. Let z be the vertex in U that is not x (there are only two). First, observe that x and z each has a chip count of $m-2$. Also observe that all vertices in V have chip counts of 0. We will now restrict the Chip Firing Game to allow only firing (recall that Lemma 3.3.4 allows use to do this) and demonstrate that no combination of firings can move chips from z to x while keeping all other chip counts greater than or equal to 0. To increase the chip count at x, we need to fire vertices in V. However, firing a vertex in V, decreases the chip count of that vertex to -2 while increasing the chip counts of x and z by 1. In order to compensate for this decrease, the vertex z must fire twice. However, firing z twice decreases its chip count by $2m$, which can never be compensated for by additional firing in V. Thus, there is no way to move any additional chips to x while maintaining an effective divisor so the maximum possible chip count at x is restricted to $m-2$. Because

$$g = mn - (m+n) + 1 > m - 2 \text{ for all } m \text{ and } n \geq 2,$$

x is not a Weierstrass point. □

Just as we did with complete graphs, we can extend the notion of a Weierstrass point to require more chips be moved to a vertex. We ask ourselves: "On which vertices in the complete bipartite graph can we concentrate the entire canonical divisor?" We address this question in Theorem 3.3.6.

3.3.6 Theorem. Let $K_{n,m}$ be the complete bipartite graph with $n > 1$ vertices in U and $m > 1$ vertices in V. The canonical divisor C_K can be concentrated on a vertex $x \in U$ if and only if $|V| = m$ divides $2g$. Likewise, C_K can be concentrated on a vertex $y \in V$ if and only if $|U| = n$ divides $2g$.

Proof. We will prove only the first statement as the second follows directly by symmetry. Observe that

$$C_K = [m-2, m-2, \ldots, m-2, n-2, n-2, \ldots, n-2]$$

and recall that, by Proposition 2.1.12,

$$\deg(C_K) = 2g - 2.$$

Let $x \in U$ and assume that x is a Weierstrass point. Then there exists a sequence of firings from C_K to

$$[2g-2, 0, 0, \ldots, 0].$$

The amount of chips that need to move to x in this firing sequence is given by

$$(2g-2) - (m-2) = 2g - m. \tag{3.3.1}$$

The number of chips moved to x in this firing sequence is the same as the number of firings that take place in V. Furthermore, each vertex in V must fire the same number of times for two reasons. First, firings in U change the chip count of all vertices in V by the same amount. Second, all vertices in V start with a chip count of $n-2$ and end with a chip count of 0. Thus, we know that

$|V| = m$ must divide $2g - m$. Since m divides m, we can conclude that m divides $2g$.

Conversely, when m divides $2g$, each vertex in V can be fired $(2g - m)/m$ times to place a total of $2g-2$ chips at x. The remaining vertices in U can then be fired, reducing their chip counts to 0 and returning the chip counts of the vertices in V to 0. $\square$

Chapter 4

Weierstrass Points and Hyperelliptic Graphs

Now that we have presented our results regarding Weierstrass points on complete and complete bipartite graphs, we will present some recent results by Baker and Norine [3]. Their work has resulted in the complete characterization of hyperelliptic graphs without Weierstrass points. In Section 4.1, we introduce the notion of harmonic morphisms on graphs. We also present Baker and Norine's Riemann-Hurwitz Theorem for graphs without proof. We use this theorem in Section 4.2 when we introduce hyperelliptic graphs and some of their defining properties. Lastly, we present Baker and Norine's theorem about hyperelliptic graphs without Weierstrass points (with proof).

4.1 Harmonic Morphisms

First, we will discuss harmonic morphisms, special maps between graphs which preserve properties associated with chip firing . We present definitions and a few lemmas which will be required to prove Baker and Norine's theorem on hyperelliptic graphs without Weierstrass points in Section 4.2.

4.1.1 Definition. Let G and G' be graphs. A function

$$\phi : V(G) \cup E(G) \to V(G') \cup E(G'),$$

written $\phi : G \to G'$ for short, is called a **morphism** if:

(i) $\phi(V(G)) \subseteq V(G')$ and

(ii) for every $x \in V(G)$ and $e \in E(G)$ such that $x \in e$ either $\phi(x) \in \phi(e)$ or $\phi(e) = \phi(x)$.

4.1.2 Definition. Let G and G' be graphs. A morphism $\phi : G \to G'$ is called **harmonic** if for each $x \in V(G)$, the quantity

$$|\{e \in E(G) \mid x \in e, \phi(e) = e'\}|$$

is the same for all $e' \in E(G')$ such that $\phi(x) \in e'$.

4.1.3 Definition. Let G and G' be graphs with $\phi : G \to G'$ a morphism between them. We define the **vertical multiplicity** of ϕ at $x \in V(G)$ by

$$v_\phi(x) = |\{e \in E(G) \mid \phi(e) = \phi(x)\}|.$$

4.1.4 Definition. Let G and G' be graphs with $\phi : G \to G'$ a harmonic morphism between them. When $|V(G')| > 1$, we define the **horizontal multiplicity** of ϕ at $x \in V(G)$ by

$$m_\phi(x) = |\{e \in E(G) \mid x \in e, \phi(e) = e'\}| \tag{4.1.1}$$

where e' is any edge in $E(G')$ with $\phi(x) \in e'$. When $|V(G')| = 1$, we define $m_\phi(x) = 0$ for all $x \in V(G)$.

4.1.5 Remark. The notion of horizontal multiplicity is only well-defined for harmonic morphisms, since the quantity in (4.1.1) does not depend on the choice of $e' \in E(G')$ by definition. This need not be true in general.

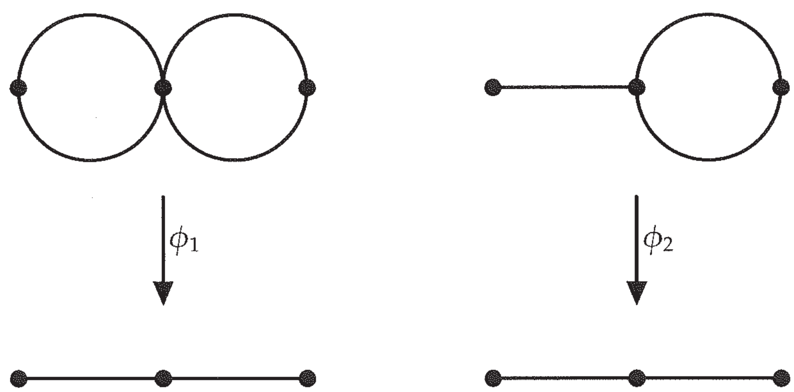

Figure 4.1.1: In this example, ϕ_1 is a harmonic morphism while ϕ_2 is a morphism which is not harmonic.

4.1.6 Definition. We say that a harmonic morphism $\phi : G \to G'$ is **non-degenerate** if $m_\phi(x) \geq 1$ for every $x \in V(G)$

4.1.7 Definition. If $|V(G')| > 1$, we define the **degree of a harmonic morphism** $\phi : G \to G'$ by the formula:

$$\deg(\phi) := |\{e \in E(G) \mid \phi(e) = e'\}|$$

for any edge $e' \in E(G')$. When $|V(G')| = 1$, we define $\deg(\phi) = 0$.

4.1.8 Lemma. ([3], Lemma 2.4) Let G and G' be connected graphs (as always). The degree of a harmonic morphism $\phi : G \to G'$ is independent of the choice of $e' \in E(G')$.

Proof. Let e'_1 and e'_2 be arbitrary adjacent edges in $E(G')$. Because e'_1 and e'_2 are adjacent, they share a vertex. Call this vertex y. Because ϕ is harmonic, we know that for each $x \in V(G)$ with $\phi(x) = y$,

$$|\{e \in E(G) \mid x \in e, \phi(e) = e'_1\}| = |\{\tilde{e} \in E(G) \mid x \in \tilde{e}, \phi(\tilde{e}) = e'_2\}|. \qquad (4.1.2)$$

Therefore, it follows quite directly that

$$\sum_{x\in\phi^{-1}(y)} |\{e \in E(G) \mid x \in e, \phi(e) = e'_1\}| = \sum_{x\in\phi^{-1}(y)} |\{\tilde{e} \in E(G) \mid x \in \tilde{e}, \phi(\tilde{e}) = e'_2\}|. \quad (4.1.3)$$

But the left hand side of (4.1.3) is $|\{e_1 \in E(G) \mid \phi(e_1) = e'_1\}|$ and the right hand side is $|\{e_2 \in E(G) \mid \phi(e_2) = e'_2\}|$. Therefore we have shown

$$|\{e_1 \in E(G) \mid \phi(e_1) = e'_1\}| = |\{e_2 \in E(G) \mid \phi(e_2) = e'_2\}|. \qquad (4.1.4)$$

Now let e'_1 and e'_2 be arbitrary in $E(G')$. Because G is connected, there is a path of adjacent edges leading from e'_1 to e'_2. Simply apply (4.1.4) to each pair of consecutive edges in this path to conclude the result. □

4.1.9 Remark. Although $\deg(\phi)$ depends only on ϕ as shown in Lemma 4.1.8, $m_\phi(x)$ does depend on $x \in V(G)$ and need not be constant throughout the graph.

4.1.10 Lemma. ([3], Lemma 2.6) For any vertex $y \in G'$, we have:

$$\deg(\phi) = \sum_{\substack{x \in V(G) \\ \phi(x) = y}} m_\phi(x)$$

Proof. Choose an edge $e' \in E(G')$ with $y \in e'$. Then

$$\sum_{x \in \phi^{-1}(y)} m_\phi(x) = \sum_{x \in \phi^{-1}(y)} \sum_{\substack{e \in \phi^{-1}(e') \\ x \in e}} 1$$
$$= |\phi^{-1}(e')| = \deg(\phi).$$

□

Up until this point in the thesis, maps have existed only between graphs. With each morphism between graphs, however, there are two induced morphisms between the set of divisors on those graphs. We present definitions of these induced morphisms now.

4.1.11 Definition. Let $\phi : G \to G'$ be a harmonic morphism. We define the **pushforward homomorphism** $\phi_* : \mathrm{Div}(G) \to \mathrm{Div}(G')$ by

$$\phi_*(D) = \sum_{x \in V(G)} D(x)(\phi(x)).$$

Similarly, we define the **pullback homomorphism** $\phi^* : \mathrm{Div}(G') \to \mathrm{Div}(G)$ by

$$\phi^*(D') = \sum_{y \in V(G')} \sum_{\substack{x \in V(G) \\ \phi(x) = y}} m_\phi(x) D'(y)(x).$$

Examples of pushforward and pullback morphisms are found in Figure 4.1.2.

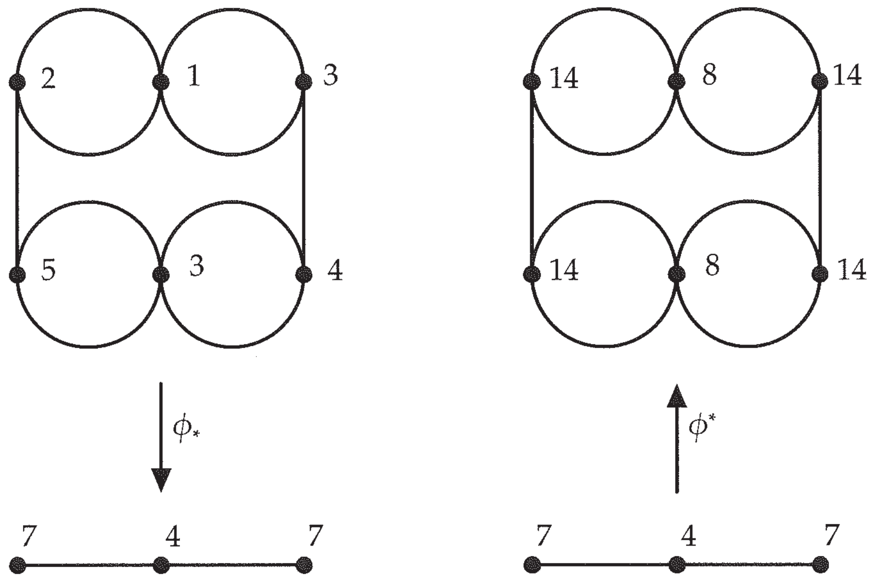

Figure 4.1.2: This figure depicts both the pushforward (ϕ_*) and the pullback (ϕ^*) of a harmonic morphisms $\phi : G \to G'$.

4.1.12 Lemma. ([3], Lemma 2.13) If $\phi : G \to G'$ is a harmonic morphism and $D' \in \mathrm{Div}(G')$, then $\deg(\phi^*(D')) = \deg(\phi) * \deg(D')$.

Proof. By the definition of ϕ^*, we know that

$$\begin{aligned}
\deg(\phi^*(D')) &= \deg\left(\sum_{y \in V(G')} \sum_{\substack{x \in V(G) \\ \phi(x)=y}} m_\phi(x) D'(y)(x) \right) \\
&= \sum_{y \in V(G')} \sum_{\substack{x \in V(G) \\ \phi(x)=y}} m_\phi(x) D'(y) \\
&= \sum_{y \in V(G')} \left(D'(y) \sum_{\substack{x \in V(G) \\ \phi(x)=y}} m_\phi(x) \right) \\
&= \sum_{y \in V(G')} \big(D'(y) \cdot \deg(\phi) \big) \text{ (Lemma 4.1.10)} \\
&= \deg(\phi) \sum_{y \in V(G')} (D'(y)) \\
&= \deg(\phi) \cdot \deg(D')
\end{aligned}$$

□

Lastly, we present selected results from [3] that will be necessary to prove Theorem 4.2.12. The proofs of these results are outside the scope of this thesis and are therefore omitted.

4.1.13 Theorem. (Riemann-Hurwitz for graphs from [3]) Let G and G' be graphs, and let $\phi : G \to G'$ be a harmonic morphism. Then:

(i) The canonical divisors on G and G′ are related by the formula

$$C_G = \phi^*(C_{G'}) + R_G,$$

where

$$R_G = 2 \sum_{x \in V(G)} (m_\phi(x) - 1)(x) + \sum_{x \in V(G)} v_\phi(x)(x).$$

(ii) If G, G' have genus g and g', respectively, then

$$2g - 2 = \deg(\phi)(2g' - 2) + \sum_{x \in V(G)} (2(m_\phi(x) - 1) + v_\phi(x)).$$

(iii) If ϕ is non-constant, we have $2g - 2 \geq \deg(\phi)(2g' - 2)$ and $g \geq g'$.

4.1.14 Definition. Let G and G' be connected graphs and let A be an abelian group. Suppose $\phi : G \to G'$ is a harmonic morphism and that $f : V(G) \to A$ and $f' : V(G') \to A$ are functions. We define $\phi_* f : V(G') \to A$ by

$$\phi_* f(y) := \sum_{\substack{x \in V(G) \\ \phi(x) = y}} m_\phi(x) f(x)$$

and $\phi^* f' : V(G) \to A$ by

$$\phi^* f' := f' \circ \phi$$

4.1.15 Lemma. ([3], Proposition 4.2) Let $\phi : G \to G'$ be a harmonic morphism.

Let $f : V(G) \to \mathbb{Z}$ and $f' : V(G') \to \mathbb{Z}$. Then,

$$\phi_*(\mathrm{div}(f)) = \mathrm{div}(\phi_* f) \tag{4.1.5}$$

and

$$\phi^*(\mathrm{div}(f')) = \mathrm{div}(\phi^* f') \tag{4.1.6}$$

4.1.16 Theorem. ([3], Theorem 4.13) Let $\phi : G \to G'$ be a non-constant harmonic morphism. Then $\phi^* : \mathrm{Jac}(G') \to \mathrm{Jac}(G)$ is injective.

4.2 Hyperelliptic Graphs

In this section we define hyperelliptic graphs. We also introduce the hyperelliptic involution which characterizes all hyperelliptic graphs. We present a collection of examples of hyperelliptic graphs (with and without Weierstrass points) as well as two important proofs from [3] regarding hyperelliptic graphs with no Weierstrass points.

4.2.1 Definition. A graph G is called **hyperelliptic** if there exists a divisor $D \in \mathrm{Div}(G)$ such that $\deg(D) = 2$ and $r(D) = 1$.

4.2.2 Definition. A graph G is called **k-edge-connected** if for every set K with of at most $k-1$ edges, $G - K$ is connected.

4.2.3 Proposition. Every graph of genus 2 is hyperelliptic.

Proof. Let G be an arbitrary graph with $g(G) = 2$. To show that G is hyperelliptic, we must find a divisor $D \in \mathrm{Div}(G)$ with $\deg(D) = 2$ and $r(D) = 1$. Let $D = C_G$, the canonical divisor of G. By Proposition 2.1.12, we know that

$$\begin{aligned} \deg(C_G) &= 2g - 2 \\ &= 2(2) - 2 \\ &= 2. \end{aligned}$$

Furthermore, by the Riemann-Roch Theorem for Graphs (Theorem 2.3.2),

$$\begin{aligned} r(D) - r(C_G - D) &= \deg(D) + 1 - g \\ r(C_G) - r(C_G - C_G) &= \deg(C_G) + 1 - g \\ r(C_G) &= 2 + 1 - 2 \\ r(C_G) &= 1. \end{aligned}$$

Therefore C_G satisfies the requirements for D and G is hyperelliptic. □

4.2.4 Proposition. The graph $G = B_n$, consisting of two vertices x and y with n edges between them, is hyperelliptic.

Proof. Let $D = (x) + (y)$. It is easy to see that $\deg(D) = 2$. Furthermore, for any divisor $E \in \mathrm{Div}(G)$, with $\deg(E) = 1$, we have

$$D - E \geq 0.$$

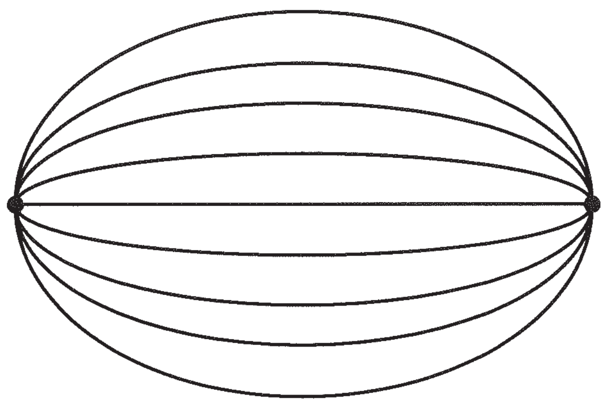

Figure 4.2.1: In this figure we present the graph B_9. This graph is hyperelliptic and has no Weierstrass points as shown in Proposition 4.2.4 and Theorem 4.2.12, respectively.

Therefore, $r(D) \geq 1$. Now let $E' = 2(x)$. It is clear that $\deg(E') = 2$. However, $D - E'$ is not equivalent to an effective divisor. Therefore $r(D) = 1$. □

4.2.5 Proposition. The graph $G = B(l_1, l_2, \ldots, l_n)$, consisting of two vertices x and y joined together by $n \geq 3$ disjoint paths of lengths $l_1, l_2, \ldots, l_n$ is hyperelliptic.

Proof. Let $D = (x) + (y)$. It is easy to see that $\deg(D) = 2$. To show that $r(D) = 1$, we will show that $|(x) + (y) - (z)| \neq \emptyset$ for all $z \in \mathring{V}(G)$. Consider one path from x to y, labeled in order by the vertices $x = z_0, z_1, z_2, \ldots, z_{l-1}, z_l = y$. If $z = x$, then $(x) + (y) - (z) = (y)$. If $z = y$, then $(x) + (y) - (z) = (x)$. These divisors are clearly effective. When $z \notin \{x, y\}$, say that $z = z_i$, the divisor $(x)+(y)-(z) = (x)+(y)-(z_i)$ is linearly equivalent to the effective divisor (z_{l-i}). This can be deduced easily from the observation that $(x) + (y) \sim (z_i) + (z_{l-i})$ for all $i \in [0, l]$. □

We now make precise the notion of a group acting on a graph.

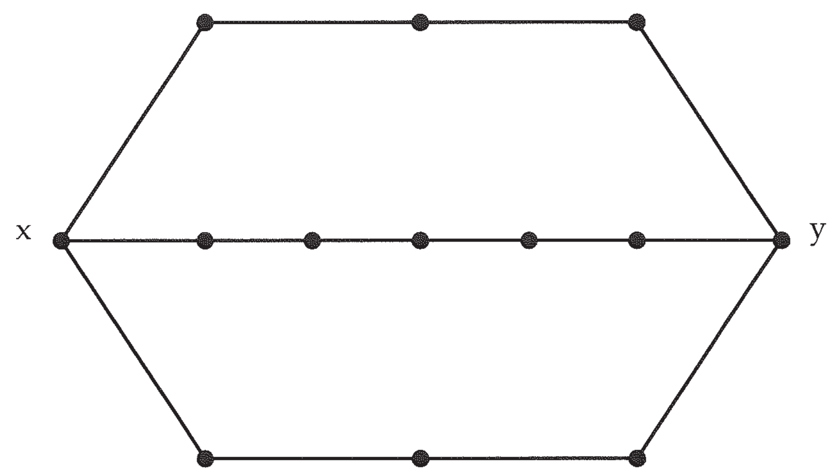

Figure 4.2.2: In this figure we present the graph $B(3,5,3)$ which is hyperelliptic and has no Weierstrass points as shown in Proposition 4.2.5 and Theorem 4.2.12, respectively.

4.2.6 Definition. Let H be a finite group acting on a graph G. Suppose we are given a homomorphism $H \to \mathrm{Aut}(G)$. We let $h \cdot x$ denote the **action** of an element $h \in H$ on an element $x \in V(G) \cup E(G)$. The **quotient graph** G/H and the canonical morphism $\pi_H : G \to G/H$ are defined by the following equivalence relation. For $x, y \in V(G) \cup E(G)$, let $x \sim_H y$ if there exists an element $h \in H$ such that $h \cdot x = y$. The vertices of G/H are defined to be the equivalence classes of $V(G)$ with respect to $\sim_H$. The edges of G/H are defined to be the equivalences classes of $E(G)$ with respect to $\sim_H$. If both endpoints of an edge are equivalent, the edge is assigned to the equivalence class of those endpoints. The **quotient morphism** $\pi : G \to G/H$ sends each edge and vertex in G to its equivalence class in G/H. If $H = \langle \phi \rangle$ is a cyclic subgroup of Aut(G), we write G/ϕ instead of G/H and $\phi^{\sim}$ instead of π_{ϕ}.

4.2.7 Definition. An automorphism i of a graph G is called an **involution** if $i \circ i$ is the identity automorphism. An involution i is said to be a **mixing involution** if for every edge $e = xy \in E(G)$ such that $i(e) = e$ we have $i(x) = y$.

We present the following introductory lemma which will be used to prove Theorem 4.2.9. The proof can be found in [3].

4.2.8 Lemma. ([3], Lemma 5.6) Let G and G' be connected graphs and let $\phi : G \to G'$ be a non-degenerate harmonic morphism of degree two. Then there is a mixing involution i of G for which $\phi = i^{\sim}$. Conversely, let $|V(G)| > 2$ and let $i : G \to G$ be a mixing involution. Then $i^{\sim}$ is a non-degenerate harmonic morphism of degree two.

The following theorem provides us with three equivalent notions of what it means for a graph to be hyperelliptic. All will be used in Theorem 4.2.12 to prove that only three types of 2-edge connected hyperelliptic graphs with no Weierstrass points exist.

4.2.9 Theorem. ([3], Theorem 5.12) For a 2-edge connected graph G of genus $g \geq 2$, the following conditions are equivalent:

(i) G is hyperelliptic.

(ii) There exists an involution $i : G \to G$ such that G/i is a tree.

(iii) There exists a non-degenerate degree two harmonic morphism ϕ from G to a tree, or $|V(G)| = 2$.

Proof. If $|V(G)| = 2$, it is easy to verify all three conditions, so we will discuss only the case where $|V(G)| \geq 3$.

(i) $\Rightarrow$ (ii) We assume that G is hyperelliptic. Therefore, there exists a divisor D with $\deg(D) = 2$ and $r(D) = 1$. For any $v \in V(G)$, $|D-(v)| \neq \emptyset$ and $\deg(D-(v)) = 1$. Since G is 2-edge connected, there is a unique w such that $(w) \sim D-(v)$. Set $i(v) = w$. For each $e = xy \in E(G)$, we define $i(e)$ as follows. If $i(x) = y$, we define $i(e) = e$. Otherwise, we apply the following procedure. Let $D_1 = (x) + (i(x))$ and let $D_2 = (y) + (i(y))$. By the definition of i, we have $D_1 \sim D \sim D_2$. Since $D_1 \sim D_2$, there exists a non-constant function $f : V(G) \to \mathbb{Z}$ such that $D_1 - D_2 = \operatorname{div}(f)$. We let $M(f)$ denote the set of vertices $z \in V(G)$ for which $f(z)$ is maximal. It is then true for any vertex $z \in M(f)$,

$$D_1(z) \geq (\operatorname{div}(f))(z) = \sum_{e'=zz' \in E(G)} (f(z) - f(z')) \geq \left|\{e' = zz' \in E(G) \mid z' \in V(G) \setminus M(f)\}\right|.$$

The above equation shows that $\deg(D_1) = 2 \geq |\delta(M(f))|$, where for a set $X \subseteq V(G)$, $\delta(X)$ denotes the set of all edges in G possessing exactly one vertex in X. This should be clear since $D_1(z) \geq \left|\{e' = zz' \in E(G) \mid z' \in V(G) \setminus M(f)\}\right|$ for all $z \in M(f)$. We also know that $|\delta(M(f))| \geq 2$ by the 2-edge-connectivity of G.

Thus, we conclude that $|\delta(M(f))| = 2$. It is also clear that both x and $i(x)$ are members of $M(f)$, since they are maximal in $\mathrm{div}(f)$ by construction. Similarly, we can conclude that f is minimized on y and $i(y)$ and therefore $y, i(y) \notin M(f)$. From this, it follows that $e = xy \in \delta(M(f))$. We now set $i(e) = e^*$, the other edge in $\delta(M(f))$ (which has exactly two elements). We now claim that i as defined is an involution of G such that G/i is a tree. It is clear from the above argument that $D_1 = (x') + (x)$. This implies that $x' = i(x)$ and by the symmetry of x and y we can easily conclude that e^* connects $i(x)$ and $i(y)$. Therefore, i is an automorphism and a mixing involution.

By Lemma 4.2.8, we know that $\phi = \pi_i$ is a degree 2 non-degenerate harmonic morphism. For every x and y in $V(G/i)$, we have:

$$\phi^*((x)) = (x) + (i(x)) \sim D \sim (y) + (i(y)) = \phi^*((y))$$

Therefore, by Theorem 4.1.16, we obtain $(x) \sim (y)$ for all $x, y \in V(G/i)$, which implies that G/i is a tree by Lemma 2.2.7.

(ii) $\Rightarrow$ (iii) Let $i : G \to G$ be an involution such that G/i is a tree (as in (ii)). Notice, that for every edge $e = xy \in E(G)$ such that $i(x) \neq y$, the set $\{e, i(e)\}$ is a cut of G, as it is the preimage of an edge in G/i. Because G is 2-edge connected, each cut must contain at least two distinct edges, implying that $e \neq i(e)$. This implies that

i is a mixing involution, which by Lemma 4.2.8 implies the existence of a non-degenerate degree 2 harmonic morphism from G to a tree, namely $\pi_i : G \to G/i$.

(iii) $\Rightarrow$ (i) Let $\phi : G \to T$ be a non-degenerate degree 2 harmonic morphism, where T is a tree. Let $y_0 \in V(T)$ be arbitrary and define the divisor $D = \phi^*((y_0))$. It is clear that D is an effective divisor of degree 2. We now show that $r(D) = 1$. Clearly, $r(D) \leq 1$, so it suffices to prove that $|D - (x)| \neq \emptyset$ for all $x \in V(G)$. Note that $(y) \sim (y')$ for every $y, y' \in V(T)$. Specifically, $\phi(x) \sim y_0$ and by Lemma 4.1.15, $D \sim \phi^*((\phi(x))) \geq m_\phi(x)(x)$. Since ϕ is non-degenerate, $m_\phi(x) > 0$ and $\phi^*((\phi(x))) = (x) + (x')$ for some $x' \in V(G)$. This implies that $|D - (x)| \neq \emptyset$ as desired. □

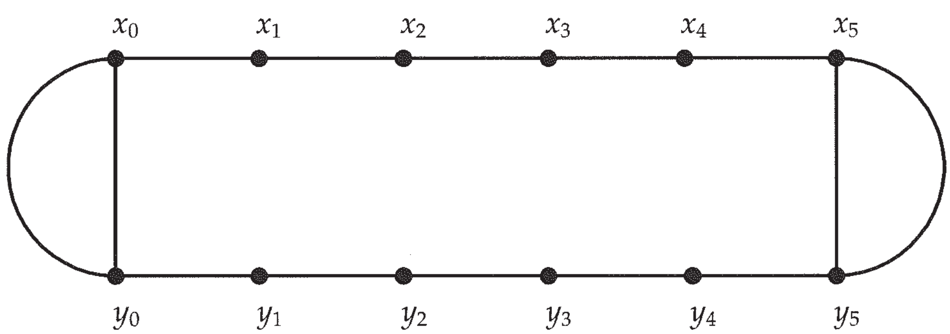

Figure 4.2.3: In this figure we present the graph $\Phi(5)$ which is hyperelliptic and has no Weierstrass points by Proposition 4.2.10 and Theorem 4.2.12,respectively.

4.2.10 Proposition. The graph $G = \Phi(l)$, which consists of two disjoint paths $P = [x_0, x_1, \ldots, x_l]$ and $Q = [y_0, y_1, \ldots, y_l]$ of length l joined by two edges between

x_0 and y_0 and another two edges between x_l and y_l, is hyperelliptic.

Proof. Consider the involution $i : G \to G$, which interchanges x_i with y_i for all i, given by $\Phi(x_i) = y_i$ and $\Phi(y_i) = x_i$. The quotient graph G/i is isomorphic to the path P, which is a tree. By Theorem 4.2.9, G is hyperelliptic. $\square$

In final preparation for Theorem 4.2.12, we state the following remark from [2].

4.2.11 Remark. For a 2-edge connected graph G, a fixed point of the hyperelliptic involution is a Weierstrass point. If $g(G) = 2$, the converse also holds.

4.2.12 Theorem. ([3], Theorem 5.26) The following are the only 2-edge-connected hyperelliptic graphs with no Weierstrass points:

(i) The graph B_n for some integer $n \geq 3$,

(ii) The graph $B(l_1, l_2, l_3)$ for some odd integers $l_1, l_2, l_3 \geq 1$, and

(iii) The graph $\Phi(l)$ for some integer $l \geq 1$.

Proof. Assume that G is a 2-edge connected hyperelliptic graph with no Weierstrass points (as in the statement of the theorem). If $|V(G)| = 2$, then G is isomorphic to the graph B_n for some $n \geq 3$, leading us to (i). In the case where $|V(G)| > 2$, we can apply Theorem 4.2.9, which guarantees the existence of a non-degenerate degree 2 harmonic morphism $\phi : G \to T$, where T is a tree with $|V(T)| > 2$. Since T is a tree, we know that $r(\phi^*((t))) = 1$ for all $t \in V(T)$. If $m_\phi(x) = 2$ for any $x \in V(G)$, then x is a Weierstrass point. This can be seen by observing that

$$r(g(x)) \geq r(2(x)) = r(\phi^*(\phi(x))) = 1.$$

Therefore, we can assume that $m_\phi(x) = 1$ for all $x \in V(G)$ so that every $t \in T$ has exactly 2 preimages under ϕ. By the Riemann-Hurwitz formula for graphs (Theorem 4.1.13), we have:

$$\sum_{x \in V(G)} v_\phi(x) = 2g - 2.$$

Consider a vertex $t \in V(T)$ with $d_t = 1$. Let $\phi^{-1}(t) = \{x, x'\}$, and let x'' be the unique neighbor of x in $V(G) \setminus \phi^{-1}(t)$ (which is well-defined since $m_\phi(x) = 1$). The number of edges incident to x is $v_\phi(x) + 1$. There is one edge connecting x to its unique neighbor x'' and $v_\phi(x)$ vertical edges connecting x to x'. Because G is 2-edge-connected, we know that $d_x \geq 2$, which implies the existence of at least one vertical edge ($v_\phi(x) \geq 1$). It follows that:

$$(v_\phi(x) + 2)(x) \sim (x) + v_\phi(x)(x') + (x'') \geq (x) + (x') = \phi^*(t).$$

Therefore $r((v_\phi(x) + 2)(x)) \geq 1$, and x is a Weierstrass point of G if $v_\phi(x) \leq g - 2$. This means that, if a vertex $x \in V(G)$ maps to the end of a branch in T ($d_{\phi(x)} = 1$), then x is a Weierstrass point provided $v_\phi(x) \leq g - 2$ Therefore, we may assume without loss of generality, that $v_\phi(x) \geq g - 1$ for all $x \in V(G)$ such that $d_{\phi(x)} = 1$. Let $k = |\{t \in V(T) \mid d_t = 1\}|$. We know that $k \geq 2$ since T is a tree. Then we can set a lower bound for the total number of vertical edges in G by

$$\sum_{x \in V(G)} v_\phi(x) \geq 2k(g-1), \tag{4.2.1}$$

since each $t \in T$ has exactly two preimages in G. But recall, from Riemann-Hurwitz (Theorem 4.1.13), we also know

$$\sum_{x \in V(G)} v_\phi(x) = 2g + 2. \tag{4.2.2}$$

By (4.2.1) and (4.2.2), we see that $2g + 2 \geq 2k(g-1)$ from which we can deduce that either $g = 2$ and $k \leq 3$ or $g = 3$ and $k = 2$. Recall that k is the number of branch ends in T. In the latter case, T is simply a path and $v_\phi(x) = 0$ for all $x \in V(G)$ with $d_{\phi(x)} \neq 1$. This can be seen by observing that for each branch end in T, there are exactly two preimages in G, yielding a total of $2k = 4$ vertices in G. For each $\hat{x} \in V(G)$ with $d_{\phi(\hat{x})} = 1$, we know that $v_\phi(\hat{x}) \geq g - 1 = 2$. Riemann-Hurwitz tells us that the total number of vertical edges in G is simply $2g+2 = 8$. Thus there are no vertical edges left for vertices $x \in V(G)$ with $d_{\phi(x)} \neq 1$. For the edges $x \in V(G)$ with $d_{\phi(x)} = 1$, it is clear that $v_\phi(x) = 2$. In this case, G is isomorphic to the graph $\Phi(|E(T)|)$ as described in (iii).

In the case where $g = 2$ and $k \leq 3$, G is isomorphic to the graph $B(l_1, l_2, l_3)$ for some integers l_1, l_2, l_3. If any of these three integers is even, then the middle vertex of the path is a Weierstrass point. Thus we reach the graph described in (ii).

It remains to be shown that the graphs described in (i), (ii), and (iii) indeed have no Weierstrass points. Suppose that $G = B_n$ with $n \geq 3$. Let x and y be the two vertices in G and consider the divisor $D = (n-1)(x) - (y)$. By Theorem 2.3.4, either $r(D) \geq 0$ or $r(\nu_P - D) \geq 0$, where

$$\nu_P = \sum_{x \in V(G)} (|\{e = xy \in E(G) \mid y <_P x\}| - 1)(x),$$

for some complete ordering $<_P$ of $V(G)$. Define $<_P$ by $x > y$. Then

$$\nu_P = (n-1)(x) + (-1)(y)$$

and $\nu - D = (0)(x) + (0)(y)$. It is clear that $r(\nu - D) \geq 0$ which implies by Theorem 2.3.4 that $r(D) = r((n-1)(x) - (y)) = -1$. Since $g = n - 1$, we can see that

$$r(g(x)) = r((n-1)(x)) \leq r((n-1)(x) - (y)) + 1 = 0,$$

which means that x is not a Weierstrass point and neither is y by symmetry.

Now suppose that $G = B(l_1, l_2, l_3)$ for some odd integers $l_1, l_2, l_3 \geq 1$ as in (ii). Because each l is odd, the hyperelliptic involution i on G can have no fixed points (since $i(z_i) = z_{l-i}$ in the language of Proposition 4.2.5). Thus, by Remark 4.2.11, we conclude that $B(l_1, l_2, l_3)$ has no Weierstrass points.

Lastly, suppose that $G = \Phi(l)$ for some $l \geq 1$. The genus of G is 3. Let the vertices of G be labeled as they are in Proposition 4.2.10. By symmetry, we need only prove that $r(g(x_i)) = r(3(x_i)) = 0$ for every integer i such that $0 \leq i \leq l$.

Suppose first that $l \leq 3i \leq 2l$. Then $3(x_i) \sim (x_0) + (x_l) + (x_{3i-l})$. Consider this linear ordering $<$ on $V(G)$:

$$y_0 < y_1 < \ldots < y_l < x_0 < \ldots < x_{3i-l-1} < x_l < \ldots < x_{3i-l+1} < x_{3i-l}$$

The divisor associated with this ordering (see Theorem 2.3.4) is

$$(x_0) + (x_l) + (x_{3i-l}) - (y_0).$$

It follows that $r(3(x_i)) = r((x_0) + (x_l) + (x_{3i-l})) = 0$. Now suppose that $3i < l$. Then we have

$$3(x_i) \sim (x_{3i}) + 2(x_0) \sim (x_{3i+1}) + 2(y_0) \sim (x_l) + (y_0) + (y_{l-3i-1}).$$

Consider this linear ordering $<$ on $V(G)$:

$$y_l < x_l < \ldots < x_0 < y_0 < \ldots < y_{l-3i-2} < y_{l-1} < \ldots < y_{l-3i} < y_{l-3i-1}$$

The divisor associated with this ordering is $(x_l) + (y_0) + (y_{l-31-1}) - (y_l)$. It again follows that $r(3(x_i)) = 0$. This is also true when $3i > 2l$ by symmetry from the case where $3i < l$. Thus $\Phi(l)$ has no Weierstrass points. $\square$

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